Geometric Representation Theory

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Introduction

Geometric Representation Theory is a study of representations of Lie groups/algebras (and their cousins) via geometric techniques. Lie groups are, as themselves, geometric objects. And geometric techniques have been used in representation theory for a long time. Borel-Weil theory, Weyl character formula, etc, are good examples.

But recent development of geometric representation theory has different flavor. Topological cohomology groups (rather than Dolbeault) play more fundamental roles. Representations appear as 'Hidden' symmetries on various moduli spaces, which have origin in theoretical physics......

§-1. ordinary representation theory of $SL(2,\mathbb{C})$

$$G = SL(2,\mathbb{C}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{C}) \;\middle|\; \det = 1 \right\}$$

$$\mathrm{Lie}\, G = \mathfrak{sl}(2,\mathbb{C}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{C}) \;\middle|\; \mathrm{tr} = 0 \right\}$$

- natural representation $\quad G \curvearrowright \mathbb{C}^2 = \mathbb{C}x \oplus \mathbb{C}y$
  (vector representation)

- $V(k) :=$ space of degree k polynomials in x & y

  It is a representation of G. (also a representation of $\mathfrak{g}$)

  - $\dim V(k) = k+1$
  - $V(k)$: irreducible
  - any finite dimensional representation is a direct sum of irreducible representations $V(k)$ $(k = 0, 1, 2, \cdots)$
    (complete reducibility and classification)

Weight spaces, highest weights

Choose a base of $\mathfrak{sl}(2,\mathbb{C})$ as $e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(They satisfy $[h,e]=2e$, $[h,f]=-2f$, $[e,f]=h$
(These relations give $\mathfrak{sl}(2,\mathbb{C})$ back.))

$$\exp(te)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+ty \\ y \end{bmatrix}$$

$$\exp(tf)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ tx+y \end{bmatrix}$$

$$x^l y^m \overset{e}{\longmapsto} l x^{l-1} y^{m+1}$$

$$\therefore\ e = \begin{pmatrix} 0 & k & & \\ & 0 & k-1 & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$$ on $V(k)$ w.r.t base $y^k, xy^{k-1}, \cdots, x^k$

$y^k \in V(k)$ is a highest weight vector: $e \cdot y^k = 0$

$$h = \begin{bmatrix} k & & & & 0 \\ & k-2 & & & \\ & & \ddots & & \\ & & & 2-k & \\ 0 & & & & -k \end{bmatrix}$$

$h \cdot (x^l y^m) = (l-m)\, x^l y^m$

eigenspace with eigenvalue $= l-m$

weight space　　weight

Weights of $V(k)$ are $k, k-2, \cdots, 2-k, -k$

highest weight $= k$　　lowest weight

$$V(k) = \bigoplus_{\substack{l=k \\ l-k:\text{even}}}^{-k} V_l(k) \quad : \text{ weight space decomposition}$$

character of $V := \sum_l \dim V_l(k)\, q^l = q^k + q^{k-2} + \cdots + q^{2-k} + q^{-k}$

— Irreducible representations (modulo isomorphisms) are determined by their highest weights.

— Finite dimensional representations (not necessarily irreducible) are determined by their characters.

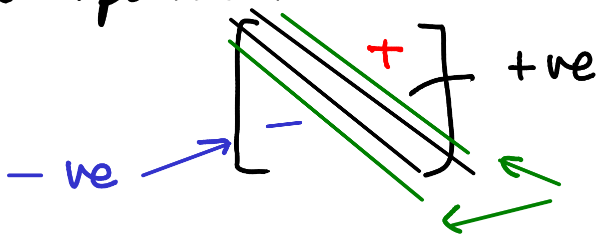
These notions depend on the choice of the base.

— Is the base or the set of generators (e, f, h) a fundamental object?

Once we pick up the diagonal h,
e & f appear as eigenvectors $[h, e] = 2e$, $[h, f] = -2f$

More general higher dimensional simple Lie algebras (e.g. $\mathfrak{sl}(n,\mathbb{C})$)
a similar construction works:
Choose a maximal commutative subalgebra (Cartan subalgebra)
(e.g. diagonal matrices $h_i = \begin{array}{cc} \begin{bmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & 1 & & \\ & & & & -1 & \\ & & & & & 0 \\ & & & & & & \ddots \\ & & & & & & & 0 \end{bmatrix} & \begin{matrix} \\ \\ \\ i \\ i+1 \\ \\ \\ \\ \end{matrix} \end{array}$)

and a decomposition roots = +ve & −ve roots



semidiagonal → generators e_i, f_i

All choices are "equivalent". Characters: Independent of the choice

But I feel that it is not a fundamental object.

However, it will play a fundamental role in constructions below.

§0. Introduction — $\mathfrak{sl}(2)$ representations

<u>Example 0</u> $V(1)$: 2-dimensional representation

$M = \{0, 1\}$ 2 points

$$H_*(M) = H_*(M, \mathbb{C}) = \mathbb{C}^2$$

$e = $ map $1 \to 0$ $\quad H_*(\{1\}) \xrightarrow{1} H_*(\{0\})$

$f = $ map $0 \to 1$ $\quad H_*(\{0\}) \xrightarrow{1} H_*(\{1\})$

$$h = \begin{cases} 1 & \text{on } H^*(\{0\}) \\ -1 & \text{on } H^*(\{1\}) \end{cases}$$

In the natural base, $e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$\therefore$ e, f, h satisfy the $\mathfrak{sl}(2)$ relations

Example 1 $V(2)$: 3-dimensional representation

$$M = T^*Gr(2) \qquad Gr(2) = \{ S \subset \mathbb{C}^2 \mid \text{subspace} \}$$
$$= \underset{\substack{\| \\ pt}}{Gr(0,2)} \sqcup \underset{\substack{\| \\ \mathbb{P}^1}}{Gr(1,2)} \sqcup Gr(2,2)$$

(cf. $Gr(1) = 2$ pts)

$$V = H_{mid}(T^*Gr(2)) \cong H_{top}(Gr(2))$$
$$= H_0(T^*Gr(0,2)) \oplus H_2(T^*Gr(1,2)) \oplus H_0(T^*Gr(2,2)) = \mathbb{C}^3$$

$$\begin{tikzcd}
H_0(T^*Gr(0,2)) \arrow[r, bend right, "f"'] \arrow[loop below, "h=2"] & H_2(T^*Gr(1,2)) \arrow[l, bend right, "e"'] \arrow[r, bend right, "f"'] \arrow[loop below, "h=0"] & H_0(T^*Gr(2,2)) \arrow[l, bend right, "e"'] \arrow[loop below, "h=-2"]
\end{tikzcd}$$

Definition of e & f

$$T^*Gr(0,2) \times T^*Gr(1,2) = pt \times T^*\mathbb{P}^1 \supset \mathbb{P}^1$$

$$\begin{tikzcd}
 & \mathbb{P}^1 \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
\underset{\substack{\| \\ pt}}{T^*Gr(0,2)} & & \underset{\substack{\| \\ T^*\mathbb{P}^1}}{T^*Gr(1,2)}
\end{tikzcd}$$

$$f(\cdot) = p_{2*}\, p_1^*(\cdot)$$
$$e(\cdot) = -\, p_{1*}\, p_2^*(\cdot)$$

$$f([T^*\underset{\substack{\| \\ pt}}{Gr(0,2)}]) = p_{2*}\, p_1^* [pt] = [\mathbb{P}^1]$$

$$e([\mathbb{P}^1]) = -p_{1*}\, p_2^* [\mathbb{P}^1] = ?$$

$$[\mathbb{P}^1] \in H_2(T^*\mathbb{P}^1) \underset{\substack{\text{Poincare} \\ \text{duality}}}{\cong} H^2_c(T^*\mathbb{P}^1)$$

cohomology with compact support

$$[\mathbb{P}^1] \mapsto e(p^*T^*\mathbb{P}^1) \in H^2_c(T^*\mathbb{P}^1)$$

$$\begin{array}{ccc} p^*T^*\mathbb{P}^1 & \longrightarrow & T^*\mathbb{P}^1 \\ \downarrow & & \downarrow \\ p: T^*\mathbb{P}^1 & \longrightarrow & \mathbb{P}^1 \end{array}$$

$$\therefore\ p_2^*([\mathbb{P}^1]) = p_2^*(e(p^*T^*\mathbb{P}^1)) = e(T^*\mathbb{P}^1) \mapsto \int_{\mathbb{P}^1} e(T^*\mathbb{P}^1) = -2[pt]$$

$$e(T^*\mathbb{P}^1) \in H^2(\mathbb{P}^1) \underset{P.D.}{\cong} H_0(\mathbb{P}^1) = \mathbb{C}$$

$$\therefore\ e([\mathbb{P}^1]) = -p_{1*}\, p_2^* [\mathbb{P}^1] = 2[pt]$$

Hence the wt 2 comes from euler # of $\mathbb{P}^1$

Generalization

- [Ginzburg]

$M = T^* Gr(k)$ $\qquad H_{mid}(M) \rightsquigarrow V(k)$

more generally $\quad M = T^*(n\text{-step flags in } \mathbb{C}^k)$

$$0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = \mathbb{C}^k$$

$\Longrightarrow$ a representation of $\mathfrak{sl}(n, \mathbb{C})$

$S^k\mathbb{C}^n$ $\quad k^{th}$ symmetric power of the natural representation

e.g. $k=1$ $\qquad 0 = F_0 \subset F_1 \subset \cdots \subset F_n = \mathbb{C}$

$\Longrightarrow n$ possibilities

$$0 = F_0 = F_1 = \cdots = F_i \subset F_{i+1} = \cdots = F_n = \mathbb{C} \qquad i = 0, 1, \cdots, n-1$$

$\therefore \quad H_{mid}(M) \cong \mathbb{C}^n$

f_i : increase dimension of F_i by 1 $\qquad (i = 1, \cdots, n-1)$

e_i : decrease $\qquad$ " $\qquad$ 1

- A similar construction of $\mathbb{C}[W]$ (W: Weyl group) was known much before, in the context of the Springer representation

$H_{mid}(T^*(G/B))$ $= H_{top}(G/B)$ (cf. $T^*\mathbb{P}^1$ gives the weight 0 space in $V(2)$)

Springer fiber : $T^*B \xrightarrow{\mu} \mathcal{N}$: nilpotent variety $\subset \mathfrak{g}$

$H_{top}(\mu^{-1}(x))$: "almost" irreducible representation of W

- [N] quiver variety

$H_{mid}(M)$: irreducible representation of a simple Lie algebra of type ADE

more generally a symmetric Kac-Moody Lie algebra

e.g. affine Lie algebra

§ 1. Convolution

M, N : oriented manifolds (compact for a moment, but noncompact later)

K : differential form on $M \times N$ gives an operator

$$\{\text{diff. forms on } N\} \longrightarrow \{\text{diff. forms on } M\}$$

$$\alpha \longmapsto K * \alpha = \int_{y \in N} K(x, y) \wedge \alpha(y)$$

If K is closed, it descends to

$$\begin{array}{ccc} H^*(N) & \longrightarrow & H^*(M) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ [\alpha] & & [K * \alpha] \end{array}$$

and it depends only on the cohomology class $[K]$ of K.

composition $\qquad \underbrace{M_1 \times M_2}_{K_{12}} \times M_3$ with $\underbrace{M_2 \times M_3}_{K_{23}}$

$$K_{12} * (K_{23} * \alpha_3) = \int_{M_2} K_{12}(x_1, x_2) \int_{M_3} K_{23}(x_2, x_3)\, \alpha_3(x_3)$$

$$= \int_{M_3} \Big\{ \underbrace{\int_{M_2} K_{12}(x_1, x_2) K_{23}(x_2, x_3)}_{\text{differential form on } M_1 \times M_3} \Big\}\, \alpha_3(x_3)$$

$$K_{12} * K_{23} := \int_{M_2} K_{12}(x_1, x_2)\, K_{23}(x_2, x_3) \qquad \text{convolution product}$$

Take $M_1 = M_2 = M_3 = M$

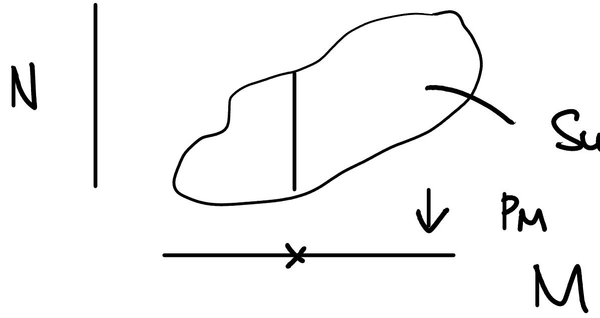
$\Rightarrow$ $H^*(M \times M)$ is an algebra and $H^*(M)$ is its representation.

But it is <u>not</u> interesting, as $H^*(M \times M) \cong H^*(M) \otimes H^*(M) \cong \mathrm{End}\, H^*(M)$

$\therefore$ $H^*(M \times M)$: matrix algebra, $H^*(M)$: vector representation

Suppose M, N : noncompact

$\int_N K(x,y) \wedge \alpha(y)$ does not converge in general

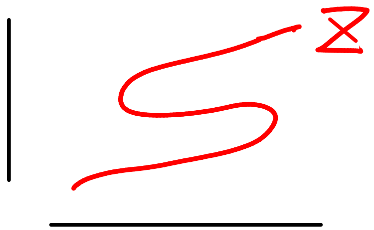


Condition

- $\operatorname{Supp} K \cap$ fiber of p_M is compact.

$\Longrightarrow \int_N K(x,y) \wedge \alpha(y)$ converges

For (co)homology



$Z \subset M \times N$ closed submanifold

condition $Z \xrightarrow[p_M]{} M$ is proper

$\Longrightarrow H^*(N) \xrightarrow{p_N^*} H^*(S) \xrightarrow{p_{M*}} H^*(M)$

$\uparrow$ well-defined

e.g. $M = N$ $\quad Z = \Delta \subset M \times M$ diagonal

$[Z]$ gives an identity operator on $H^*(M)$

Rem Poincaré duality for (oriented) noncompact manifold

$$H^*_c(M) \cong H_{\dim_{\mathbb{R}} M - *}(M) \qquad \alpha \mapsto \alpha \cap [M]$$

$$H^*(M) \cong H^{BM}_{\dim_{\mathbb{R}} M - *}(M)$$

Borel–Moore homology = homology of locally finite chains

$$f : M \to N \quad \text{proper} \Rightarrow \quad H^{BM}_*(M) \to H^{BM}_*(N)$$

$T^*Gr(m,k) \times T^*Gr(m+1,k) \supset$ conormal bundle to the correspondence

$$T^*Gr(m,k) = \{ (\xi_1, S_1) \mid \xi_1 : \mathbb{C}^k/S_1 \to S_1 \}$$

$$\{ (S_1, S_2) \mid S_1 \subset S_2 \} \subset Gr(m,k) \times Gr(m+1,k)$$

$$= \left\{ (S_1, S_2, \xi) \;\middle|\; \begin{array}{l} S_1 \subset S_2 \\ \xi : \mathbb{C}^k/S_2 \to S_1 \end{array} \right\}$$

In general, we need to consider the case when Z has singularities

e.g. M, N : cpx mfd, Z : analytic subvariety in $M \times N$

Poincaré duality : $H^{BM}_*(S) = H^{\dim M - *}(M \times N, M \times N \setminus Z)$ fund. class $[Z]$ is defined

Remark symplectic geometry

M, N : symp. mfd $\supset Z$ lagrangian

quantization

$\rightsquigarrow$ $\mathcal{H}(M\times N) = \mathcal{H}(M)\otimes\mathcal{H}(N)$ Hilbert space $\ni \mathcal{H}(Z)$

$\simeq \mathrm{Hom}(\mathcal{H}(M), \mathcal{H}(N))$ operator

More concretely, symp / lagr. assumption gives up a natural degree convention

$$[Z] \in H_{\frac{1}{2}(\dim_{\mathbb{R}} M + \dim_{\mathbb{R}} N)}(Z)$$

$$\Rightarrow H^{BM}_{\frac{1}{2}\dim_{\mathbb{R}} N + *}(N) \xrightarrow{[Z]*} H^{BM}_{\frac{1}{2}\dim_{\mathbb{R}} M + *}(M)$$

check

$$\begin{tikzcd} & Z \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ M & & N \end{tikzcd}$$

$$H^{BM}_{\frac{1}{2}\dim_{\mathbb{R}} N + *}(N)$$

$$\cong$$

$$H^{\frac{1}{2}\dim_{\mathbb{R}} N - *}(N) \xrightarrow{p_2^*} H^{\frac{1}{2}\dim_{\mathbb{R}} N - *}(Z)$$

$$\cong$$

$$H^{BM}_{\frac{1}{2}\dim_{\mathbb{R}} M + *}(Z) \xrightarrow{p_{1*}} H^{BM}_{\frac{1}{2}\dim_{\mathbb{R}} M + *}(M)$$

§2. Symmetric products and 1^D Heisenberg algebras

X : oriented C^∞-manifold, compact for simplicity

$H_*(X) = H_{even}(X)$ for simplicity

$S^n X = \underbrace{X \times \cdots \times X}_{n} / \mathfrak{S}_n$: n^{th} symmetric product of X

not manifold, but P.D. holds

$\underline{Q}$. $\displaystyle\bigoplus_{n=0}^{\infty} H_*(S^n X) = ?$

$\underline{A}$. 1^D Heisenberg algebra, modelled on $H_*(X)$

$\alpha \in H_*(X) \rightsquigarrow P(\alpha), Q(\alpha)$: $\bigoplus_n H_*(S^n X) \circlearrowleft$

annihilation creation operators

$P(\alpha)$ decrease n by 1

$Q(\alpha)$ increase n by 1

relation $[P(\alpha), Q(\beta)] = (\alpha.\beta) \cdot \mathrm{id}$

where $(\alpha.\beta) = \int_X \alpha \cup \beta$

Construction

$$X \times S^n X \times S^{n+1} X \supset Z = \{ (x, \sigma_1, \sigma_2) \mid \sigma_2 = x + \sigma_1 \}$$

$$\dim Z = (n+1) \dim X = \frac{1}{2} \dim (X \times S^n X \times S^{n+1} X)$$

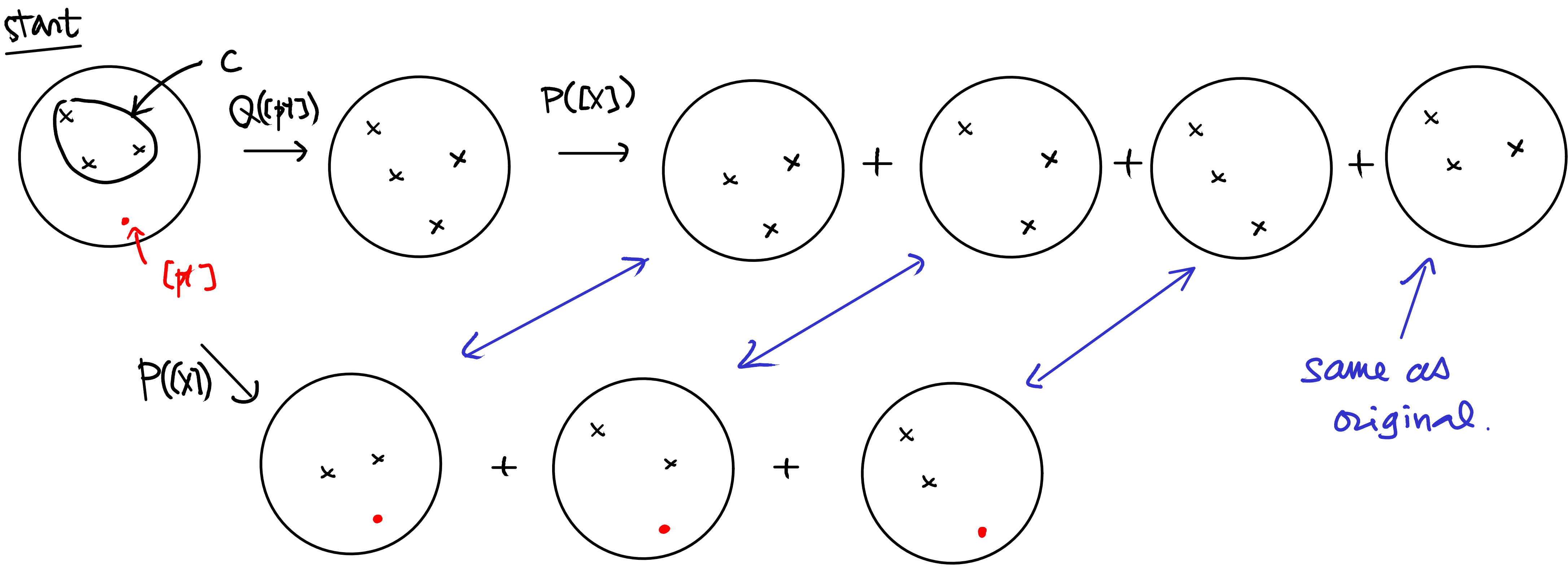
$$\begin{tikzcd}
 & Z \arrow[r, "p_X"] \arrow[dl, "p_1"'] \arrow[d, "p_2"] & X \\
S^n X & S^{n+1} X &
\end{tikzcd}$$

$$\alpha \in H_{\frac{1}{2}+*}(X) \rightsquigarrow p_X^*(\alpha) \in H_{\frac{1}{2}+*}(Z)$$

$$Q(\alpha) := p_{2*}(p_1^*(\cdot) \cap p_X^*(\alpha)) : H_{\frac{1}{2}+*}(S^n X) \longrightarrow H_{\frac{1}{2}+*+*}(S^{n+1} X)$$

$$P(\alpha) := p_{1*}(p_2^*(\cdot) \cap p_X^*(\alpha)) : \longleftarrow$$

Check $\quad [P([X]), Q([pt])](c) = c \quad \forall c \in H_*(S^n X)$


start
C
Q([p])
P([X])
[p]
P([X])
same as
original.